DETERMINANT

Part I

1 WHAT IS A DETERMINANT ?

Every square matrix can be associated to a number which is known as a Determinant.

If A ⟶ square matrix

|A| or det A or Δ ⟶ denotes the determinant of A

Columns : C_1 C_2 C_3

$$\text{Rows: } \begin{matrix} R_1 \to \\ R_2 \to \\ R_3 \to \end{matrix} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

⟶ is a determinant of order '3'.

All entries (1,2,3,4,5,6,7,8,9) are called elements of the determinant.

2 SUBMATRIX

A matrix obtained by deleting some rows or columns is said to be a submatrix.

$$\text{If } A = \begin{bmatrix} a & b & c & d \\ x & y & z & w \\ p & q & r & s \end{bmatrix}; \quad \text{Then} \begin{bmatrix} a & c \\ x & z \\ p & r \end{bmatrix}, \quad \begin{bmatrix} a & b & d \\ p & q & s \end{bmatrix}, \quad \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}$$

are all submatrices of A.

3 MINORS & COFACTORS

MINORS : are defined as the determinant of the sub matrix obtained by deleting i^{th} row and j^{th} column of the determinant (let determinant be Δ) **Denoted by M_{ij}**

COFACTOR : denoted by C_{ij} and is defined by **$C_{ij} = (-1)^{i+j} M_{ij}$**

4 HOW TO FIND THE DETERMINANT ?

Matrix should be square matrix of order greater then 1, let $A = [a_{ij}]_{n \times n}$

Determinant of A is defined as sum of products of elements of any one row (or one column) with corresponding cofactors.

$$A_{ij} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$\Rightarrow |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ (using 1^{st} row)

$$\text{or } |A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

PROPERTIES OF DETERMINANTS

Part II

P - 1 The value of a determinant remains unaltered, if the rows & columns are interchanged.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D' \Rightarrow D \;\&\; D' \text{ are transpose of each other.}$$

P - 2 If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D' = -D$$

P - 3 If a determinant has any two rows (or columns) identical, then its value is zero.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ then it can be verified that } D = 0.$$

P - 4 If all the elements of any row (or column) be multiplied by the same number then the determinant is multiplied by that number.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D' = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D' = KD$$

P - 5 If each element of any row (or column) can be expressed as a sum two terms then the determinant can be expressed as the sum of two determinants.

$$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

P - 6 The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column).

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D' = \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3+na_2 & b_3+nb_2 & c_3+nc_2 \end{vmatrix}. \text{ Then } D' = D$$

Note :- While applying this property atleast one row (or column) must remain unchanged.

Part III

DETERMINANT : CRAMER'S RULE

Simultaneous linear equations involving three unknowns x, y and z

$$a_1x + b_1y + c_1z = d_1 \ldots\ldots\ldots\ldots(i)$$
$$a_2x + b_2y + c_2z = d_2 \ldots\ldots\ldots\ldots(ii)$$
$$a_3x + b_3y + c_3z = d_3 \ldots\ldots\ldots\ldots(iii)$$

The solution for the above system of linear equations

$$x = \frac{D_1}{D} \; ; \; y = \frac{D_2}{D} \; ; \; z = \frac{D_3}{D}$$

where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} ; \; D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \quad D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} ; \; D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Nature of the Solutions

- When $D = 0$
  - At least one of D_1, D_2, D_3 is non-zero ⇒ Equations are inconsist-ent & have no solution
  - All $D_1 = D_2 = D_3 = 0$ ⇒ Equations are consistent & have infinite solution
- When $D \neq 0$
  - All least one of D_1, D_2, D_3 is non-zero ⇒ Equations are consistent & have unique solution
  - All $D_1 = D_2 = D_3 = 0$ ⇒ Equations are consistent & have a trivial solution

➡ If a given system of linear equations have only zero solution for all its variables then the given equations are said to have trivial solution.

➡ If a system of linear equations (in two variables) have definite & unique solution, then they represent intersecting lines.

➡ If a system of linear equations (in two variables) have no solution, then they represent parallel lines.

➡ If a system of linear equations (in two variables) have infinite solutions, then they represent Identical lines.